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Corner transfer matrix of generalized free-Fermion vertex systems

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Abstract. The Hamiltonian limit of the corner transfer matrix (CTM) of a generalized free-Fermion vertex system of finite size leads to a quantum spin Hamiltonian of the particular form

$$\mathcal{H}_N = -\sum_{n=1}^{N-1} \left\{ n \left(\sigma_n^x \sigma_{n+1}^x + \lambda \sigma_n^y \sigma_{n+1}^y + h(\sigma_n^z + \sigma_{n+1}^z) \right) \right\} .$$

Diagonalization may be achieved for all pairs of parameters (λ, h) with the use of some new elliptic polynomials which extend the class of special polynomials known so far in the context of CTM.

1. Introduction

Recently we have studied the corner transfer matrix (CTM) of a free-Fermion eight-vertex system at criticality [1]. This investigation has been part of a series of studies devoted to special cases in the parameter space of this model where an explicit solution can be found for the diagonalization of such CTMs. The CTM is an interesting object in the statistical mechanics of two-dimensional lattice models which was introduced by Baxter in the 70's (see [2] for a review and references to the original publications), and with which he was able to compute the mean magnetization (or polarization) in exactly solvable models, such as the eight-vertex model. Baxter has shown then that in the infinite limit of the two-dimensional lattice the spectrum of the 'generator' of the CTM is an equidistant spectrum in some domain of the coupling constants. This result has been largely confirmed by numerous computations in vertex models of the RSOS type [3].

The pecularity of this spectrum has aroused interest in the last decade for another discovery in critical phenomena which has a similar structure. The hypothesis of conformal invariance of critcal systems advocated by Belavin *et al* [4] leads to the conclusion that the generators of row-to-row transfer matrices [5] as well as of CTM [6] of finite-size critical systems also have equidistant spectra. It is therefore natural to ask whether there exists a relation between these observations. In an attempt to clarify this question we have undertaken a systematic study of the CTM of the simplest soluble system, the free-Fermion system (or its equivalent vertex model). We shall usually place ourselves at arbitrary temperature T and consider finite but large systems of size N. Therefore both limits $T \to T_c$ and $N \to \infty$ can be taken independently.

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Let us summarize what has been obtained so far in the study of the CTM of the free-Fermion system. The most general free-Fermion system depends on two parameters: the anisotropy parameter λ, which measures temperature, and the reduced external magnetic field h [7]. From the standard method of Lieb *et al* [8] it is known that the CTM of a generalized free-Fermion system may be diagonalized by diagonalizing an associated matrix obtained from the eigenvalue equation of the generator of the CTM. This matix has the peculiar feature that its eigenfunctions are special polynomials defined by recursion relations. In [9] we have solved the simplest case

- $\lambda = 1$ and $h = 0$

which is also a critical line. The polynomials obtained are particular cases of Meixner–Pollaczek polynomials. In [10] we have considered the case of

- arbitrary λ and $h = 0$

corresponding to an Ising model and found two types of Carlitz polynomials of imaginary argument. Then several other cases are solved in [7]:

- $\lambda = 1$ and h arbitrary

where Pollaczek and Gottlieb polynomials are obtained:

- $\lambda = h^2$, the disorder line [11]

where Gottlieb polynomials are found. It is remarkable that the third class of Carlitz polynomials is also found in the CTM of the discrete Gaussian model [12]. Lately we have also obtained the generalized Pollaczek polynomials in a free-Fermion vertex model with a line of defects [13]. Up to now the polynomials encountered are all orthogonal polynomials already known in the mathematical literature. Recently we have tackled other regimes in the parameter space (λ, h) where the orthogonality property is not known. First we have studied in [1] the polynomials associated with the critcal line

$$\lambda = 2h - 1$$

and in this paper we shall deal with the general case

$$\lambda > h^2$$

where neither λ nor h are restricted to satisfy any equation. The remaining regime of the parameter space, $\lambda < h^2$, where additional mathematical complexities arise, will be treated in a forthcoming publication.

As the reader may have noticed, the CTM of free-Fermion systems is a simple device of statistical mechanics which introduces the special polynomials. The spectrum of a CTM of size N is 'essentially' given by the zeros of a polynomial of size N, when N is sufficiently large. Hence, as one may guess, the distribution of zeros tends to be a uniform distribution as $N \to \infty$. This fact confirms the findings of Baxter and verifies naturally the predictions of conformal invariance in critical systems.

The new feature in this paper is the appearance of new 'elliptic' polynomials which generalize the only known types of elliptic polynomials discovered by Carlitz [14]. Some properties of these polynomials including the asymptotic distribution of their zeros, hence the eigenvalue spectrum of the system, will be given.

Physically the 'time' generator of such free-Fermion eight-vertex problems is simply the anisotropic XY quantum spin chain in a magnetic field h. The counterpart of this using the CTM approach is the following generator for a finite chain of N sites:

$$L_0 = \sum_{n=1}^{N-1} \{n(\sigma_n^x \sigma_{n+1}^x + \lambda \sigma_n^y \sigma_{n+1}^y) + h(2n-1)\sigma_n^z\} + h(N-1)\sigma_N^z \tag{1}$$

where λ is the anisotropy parameter essentially describing the temperature. Note the linear increase of the coupling strength along the chain. The problem of the simple chain, i.e. without that specific linear increase of the couplings, has been fully solved long ago [15], but the generator L_0 has not yet, to our knowledge, been solved explicitly for general values of λ and h.

The paper is organized as follows. In section 2 we outline the method employed for calculating analytically the eigenvalues of the generator L_0 which is based on the introduction of generating functions for the components of the eigenvectors of L_0. The recursion relations derived for the components of the eigenvectors are then equivalent to a set of coupled first-order differential equations which are solved formally in section 3. In the solution we encounter an elliptic integral. We need to appropriately parametrize and invert this elliptic integral, which is done in section 4. This leads us to expressions for the generating functions in section 5 which are used in section 6 to derive explicitly the components of the eigenvectors as elliptic polynomials from Cauchy's theorem. In section 7 finally we obtain an integral representation for the components of the eigenvectors that can be used to calculate asymptotically, for a large system size N, the eigenvalues of L_0. Section 8 summarizes our findings and gives an outlook on open problems. In the appendix we give some details for certain limiting cases of the analysis of the main text.

2. Method for diagonalizing L_0

The standard method we adopt here is that of Lieb *et al* [8] which consists of rewriting L_0 in terms of Fermion operators. Then one is left with the diagonalization of two non-commuting matrices $(A - B)$ and $(A + B)$ in the language of Lieb *et al*. If we denote the components of the eigenvectors ψ and ϕ by $\psi = (\psi_1, \ldots, \psi_N)$ and $\phi = (\phi_1, \ldots, \phi_N)$, we have two recursion relations coupling the components ψ_n and ϕ_n:

$$(n-1)\psi_{n-1} + n\lambda\psi_{n+1} - h(2n-1)\psi_n = \varepsilon\phi_n \tag{2}$$

$$\lambda(n-1)\phi_{n-1} + n\phi_{n+1} - h(2n-1)\phi_n = \varepsilon\psi_n\,. \tag{3}$$

For the end components, $n = N$, of the finite chain we have only

$$(N-1)\psi_{N-1} - h(N-1)\psi_N = \varepsilon\phi_N \tag{4}$$

$$\lambda(N-1)\phi_{N-1} - h(N-1)\phi_N = \varepsilon\psi_N\,. \tag{5}$$

Using the recursion relations (2) and (3), the equations for the end components (4) and (5) are equivalent to

$$\phi_{N+1} = h\phi_N \tag{6}$$

$$\lambda\psi_{N+1} = h\psi_N \tag{7}$$

which resemble periodic boundary conditions. Note that the components ϕ_{N+1} and ψ_{N+1} are only defined through these equations.

The method employed is simple in principle: it consists of finding an expression for ψ_n and ϕ_n, then substituting into (4) and (5), thus obtaining the eigenvalues ϵ. However, as we are only interested in large values of N, only an asymptotic expression will be necessary. For the cases known so far [7], the ψ_n and ϕ_n are expressible in terms of known special polynomials.

We thus expect that for general values of λ and h they are also polynomials of the elliptic type which are seen in the case of the doubled Ising model [10]. In special cases where λ and h were chosen to satisfy particular conditions it was previously possible to identify the recursion relations (2) and (3), after some trivial transformations, with the recursion relations for some known polynomials. Here, as in another case discussed recently [13], we introduce the generating functions as formal power series in a parameter t with the components of the eigenvectors as coefficients

$$\psi(t) = \sum_{n=1}^{\infty} t^{n-1}\psi_n \tag{8}$$

$$\phi(t) = \sum_{n=1}^{\infty} t^{n-1}\phi_n\,. \tag{9}$$

The recursion relations (2) and (3) are then equivalent to a set of coupled first-order differential equations for the generating functions

$$(t^2+\lambda-2ht)\psi' + (t-h)\psi = \varepsilon\phi \tag{10}$$

$$(\lambda t^2+1-2ht)\phi' + (\lambda t-h)\phi = \varepsilon\psi\,. \tag{11}$$

From the explicit solutions $\psi(t)$ and $\phi(t)$ of these differential equations with the given initial conditions one may then extract ϕ_n and ψ_n by using the Cauchy theorem. Then for large N the boundary conditions (6) and (7) determine the spectrum of L_0 as in previous works on this series [1, 7]. Since the presence of two parameters λ and h complicates the mathematical working considerably, we shall proceed step by step in presenting the solution. The central problem at hand is simply the parametrization by elliptic functions of the solution of the coupled set of differential equations (10) and (11).

3. Formal solution of the differential equations—generating functions

Before writing down the formal solution of the differential equations (10) and (11), we note that these equations, as well as the recursion relations for ψ_n and ϕ_n (2), (3), remain globally invariant under the combined transformations

$$\lambda \to \lambda^{-1} \qquad h \to \lambda^{-1}h \qquad \text{and} \qquad \epsilon \to \lambda^{-1}\epsilon$$

followed by

$$\psi \to \phi \qquad \text{and} \qquad \phi \to \psi \qquad (\psi_n \to \phi_n \qquad \text{and} \qquad \phi_n \to \psi_n)\,.$$

This property allows us to restrict our study to the domain S defined by

$$0 < \lambda < 1 \qquad \text{and} \qquad 0 < h < 1\,.$$

Using this symmetry relation, integrating factors can be found [16] and, as in [1], the general solutions of the differential equations take the form of Meixner's generating functions [17] (up to constants)

$$\psi(t) \propto f(t)\exp(\varepsilon w(t;\lambda,h)) \tag{12}$$

$$\phi(t) \propto g(t)\exp(\varepsilon w(t;\lambda,h)) \tag{13}$$

where

$$f(t) = (t^2 + \lambda - 2ht)^{-1/2} \qquad g(t) = (\lambda t^2 + 1 - 2ht)^{-1/2} \tag{14}$$

and $w = w(t; \lambda, h)$ is the two-parameter elliptic integral

$$w(t; \lambda, h) = \int_{t_0}^{t} \mathrm{d}x \, f(x) g(x) \,. \tag{15}$$

The lower bound of the integral will be chosen so as to agree with known results. Note that for $h = 0$ one recovers the case of the doubled Ising model studied in [10]. Generating functions of the type (12) and (13) are called generating functions of the Meixner type by Chihara [18]. Thus it appears that (12) and (13) represent perhaps the most general case known so far. The main problem here is the inversion of the two-parameter elliptic integral $w(t)$, (15), and then the representation of the ψ_n and ϕ_n as contour integrals.

4. Parametrization and inversion of the elliptic integral $w(t)$

An essential step in the explicit solution consists in an appropriate parametrization of the elliptic integral $w(t)$. We shall follow here the procedure given by Greenhill [19]. Both polynomials $N(t) \equiv t^2 + \lambda - 2ht$ and $D(t) \equiv \lambda t^2 + 1 - 2ht$ appearing in $w(t)$ have the same discriminant $\Delta = h^2 - \lambda$. The zeros of $N(t)$ in the parameter range $0 < \lambda < 1$ and $0 < h < 1$, the region S, are

$$\gamma = h + \sqrt{\Delta} \qquad \delta = h - \sqrt{\Delta} \tag{16}$$

whereas $D(t)$ has zeros at

$$\alpha = \frac{\gamma}{\lambda} \qquad \beta = \frac{\delta}{\lambda} \,. \tag{17}$$

The square S is divided into two regions by the disorder line $\lambda = h^2$ [7], separating oscillating from monotonous behaviour of the correlation functions [11].

In the region S_1 ($\lambda > h^2$) the zeros are pairwise complex conjugate, whereas in the region S_2 ($\lambda < h^2$) *the zeros are all real and ordered according to* $-\infty < \delta < \gamma < \beta < \alpha < \infty$.

The inversion of the elliptic integral in the region S_2 is more involved and we shall defer its study to a subsequent publication. In the remainder of this paper we shall be only concerned with the region S_1.

The region S_1 is limited by three boundaries which contain known results [7]:

- $h = 0$—the doubled Ising model, viewed as a free-Fermion eight-vertex model [10]: ψ_n and ϕ_n are Carlitz elliptic polynomials of imaginary arguments.
- $\lambda = 1$—isotropic case in the presence of a magnetic field h. The solutions ψ_n are given in terms of Meixner Pollaczek polynomials.
- $\lambda = h^2$—the disorder line, where the solutions ψ_n are expressed in terms of Gottlieb (Meixner polynomials of the first kind) polynomials.

Note that on these three boundaries the three types of polynomials are all orthogonal polynomials, and their recursion relations are always reducible to tridiagonal form. The ψ_n and ϕ_n studied here provide an interpolation between the three classes of polynomials and are presumably also orthogonal polynomials.

The elliptic integral $w(t)$ may be inverted in a standard way: one may give t as a function of w following the example given in the book of Greenhill [19, section 70]. In the following we outline Greenhill's procedure.

Through the change of variables

$$y(t) = \frac{N(t)}{D(t)} = \frac{t^2 + \lambda - 2ht}{\lambda t^2 + 1 - 2ht} \tag{18}$$

the elliptic integral is transformed from its Jacobian form $w(t)$ into the Weierstraß form

$$w(y) = \frac{1}{\sqrt{\lambda - h^2}} \int_{y_0}^{y} \frac{\mathrm{d}x}{\sqrt{4x(y_1 - x)(x - y_2)}} \tag{19}$$

where $y_0 = y(t_0)$ and y_1 and y_2 are the maximum and minimum values of $y(t)$, respectively, at the points t_1 and t_2 related by

$$h^2(1 - y_{1,2})^2 = (\lambda - y_{1,2})(1 - \lambda y_{1,2}) \qquad y_1 = y_2^{-1} \tag{20}$$

$$t_{1,2}^2 - \frac{1+\lambda}{h} t_{1,2} + 1 = 0 \qquad t_1 = t_2^{-1} \tag{21}$$

and

$$y_2 = \frac{1}{2(\lambda - h^2)} \left\{(1 + \lambda^2 - 2h^2) - (1 - \lambda)\sqrt{(1+\lambda)^2 - 4h^2}\right\} < y_1 . \tag{22}$$

Then from the standard Weierstraß form one may solve the inversion problem in three different ways:

$$\begin{aligned} w(y) &= \sqrt{\frac{y_2}{\lambda - h^2}} \operatorname{sn}^{-1}\left(\sqrt{\frac{y_1 - y}{y_1 - y_2}}, \kappa\right) \\ &= \sqrt{\frac{y_2}{\lambda - h^2}} \operatorname{cn}^{-1}\left(\sqrt{\frac{y - y_2}{y_1 - y_2}}, \kappa\right) \\ &= \sqrt{\frac{y_2}{\lambda - h^2}} \operatorname{dn}^{-1}(\sqrt{y_2 y}, \kappa) \end{aligned} \tag{23}$$

in terms of the Jacobian elliptic functions $\operatorname{sn}(v, \kappa)$, $\operatorname{cn}(v, \kappa)$ and $\operatorname{dn}(v, \kappa)$ of argument v and modulus κ. In particular, the modulus κ is given by $\kappa^2 = 1 - y_2^2$, thus y_2 is the complementary modulus of κ. Conversely, for later use one can extract from (23) the quantities

$$\begin{aligned} y_1 - y &= (y_1 - y_2)\operatorname{sn}^2(wq, \kappa) \\ y - y_2 &= (y_1 - y_2)\operatorname{cn}^2(wq, \kappa) \\ y &= \frac{1}{y_2}\operatorname{dn}^2(wq, \kappa) \end{aligned} \tag{24}$$

with $q = \sqrt{(\lambda - h^2)/y_2}$. To obtain t as a function of w, i.e. to invert the elliptic integral (15), we use equation (102) of Greenhill [19]

$$y_1 - y = \frac{(\lambda y_1 - 1)(t_1 - t)^2}{\lambda t^2 + 1 - 2ht} \tag{25}$$

$$y - y_2 = \frac{(1 - \lambda y_2)(t - t_2)^2}{\lambda t^2 + 1 - 2ht} . \tag{26}$$

By dividing out these equations and using the first two equations of (24), as well as $y_1 y_2 = 1$ and $t_1 t_2 = 1$, we obtain the following form for $t = t(w)$:

$$t(w) = \frac{t_1\sqrt{\lambda - y_2}\,\mathrm{cn}(qw,\kappa) + t_2\sqrt{y_2(1-\lambda y_2)}\,\mathrm{sn}(qw,\kappa)}{\sqrt{\lambda - y_2}\,\mathrm{cn}(qw,\kappa) + \sqrt{y_2(1-\lambda y_2)}\,\mathrm{sn}(qw,\kappa)}\,. \tag{27}$$

Thus in the case of the two parameters λ and h the inversion of (15) yields a rational function of the Jacobi elliptic functions sn and cn. However, in order to agree with the known results on the boundary line $h = 0$, where we have a canonical elliptic integral of Legendre form and hence t is simply proportional to a Jacobi elliptic $\mathrm{sn}(v, k)$ function, as was always the case in previous studies [7] on special lines where one had therefore only one parameter, we transform to imaginary argument and complementary modulus y_2 ($\kappa^2 + y_2^2 = 1$) with the transformation formulae (Jacobi's imaginary transformation)

$$\mathrm{sn}(qw,\kappa) = -\mathrm{i}\frac{\mathrm{sn}(\mathrm{i}qw,\kappa')}{\mathrm{cn}(\mathrm{i}qw,\kappa')} \qquad \mathrm{cn}(qw,\kappa) = \frac{1}{\mathrm{cn}(\mathrm{i}qw,\kappa')}\,. \tag{28}$$

Thereby we obtain

$$t = \frac{t_1 - \mathrm{i}\sqrt{y_2}\,\mathrm{sn}(\mathrm{i}qw, y_2)}{1 - \mathrm{i}t_1\sqrt{y_2}\,\mathrm{sn}(\mathrm{i}qw, y_2)}\,. \tag{29}$$

At $h = 0$ one may directly invert the elliptic integral of Jacobian form

$$w(t) = \int_0^t \frac{\mathrm{d}x}{\sqrt{(\lambda x^2 + 1)(x^2 + \lambda)}}$$

to obtain

$$t(w) = \sqrt{\lambda}\frac{\mathrm{sn}(w,\lambda')}{\mathrm{cn}(w,\lambda')} \qquad \lambda' = \sqrt{1-\lambda}\,.$$

By transformation to imaginary argument and complementary modulus, this yields

$$t = -\mathrm{i}\sqrt{\lambda}\,\mathrm{sn}(\mathrm{i}w, \lambda)\,. \tag{30}$$

Of course we require that (29) reduces to (30) in the limit $h \to 0$. To achieve this we shift the argument in (29) by K', the complete elliptic integral of the complimentary modulus, i.e. the quarter period in the imaginary direction of the Jacobian elliptic functions: $w = -u + w_0$ with $qw_0 = K'(y_2)$, which transforms the function sn according to

$$\mathrm{sn}(v + \mathrm{i}K', k) = \frac{1}{k\,\mathrm{sn}(v,k)}\,.$$

Through this last transformation we arrive at

$$t = \frac{t_2 - \mathrm{i}\sqrt{y_2}\,\mathrm{sn}(\mathrm{i}qu, y_2)}{1 - \mathrm{i}t_2\sqrt{y_2}\,\mathrm{sn}(\mathrm{i}qu, y_2)}\,. \tag{31}$$

Basically this result means that we have to choose the lower integration bound in (15) in such a way that (31) holds. For $h \to 0$ (i.e. $y_2 \to \lambda$ and $t_2 \to 0$) equation (31) now agrees with (30). From now on we shall use this condition at $h \to 0$ as a reference which will also fix the constants in the generating functions (12) and (13).

5. Expressions for the generating functions

From (26) we have for the prefactor of the generating function $\phi(t)$

$$\sqrt{\lambda t^2 + 1 - 2ht} = \sqrt{\frac{1-\lambda y_2}{y - y_2}}(t - t_2)\,.$$

Using the second equation of (24) and performing both transformations described above consecutively, i.e. Jacobi's imaginary transformation and the transformation according to the shifted argument, we obtain, together with (31), to evaluate $(t - t_2)$

$$\frac{1}{\sqrt{\lambda t^2 + 1 - 2ht}} = \sqrt{\frac{1 - y_2^2}{1 - \lambda y_2}}\,\frac{1 - \mathrm{i}t_2\sqrt{y_2}\,\mathrm{sn}(\mathrm{i}qu, y_2)}{(1 - t_2^2)\,\mathrm{dn}(\mathrm{i}qu, y_2)}\,.$$

For $h \to 0$ this expression has the correct limit [10]

$$\lim_{h\to\infty} \frac{1}{\sqrt{\lambda t^2 + 1 - 2ht}} = \frac{1}{\mathrm{dn}(\mathrm{i}qu, \lambda)}\,.$$

Hence we obtain the generating function of the ϕ_n as

$$\phi(t) = \frac{\mathrm{e}^{w\epsilon}}{\sqrt{\lambda t^2 + 1 - 2ht}} = \sqrt{\frac{1 - y_2^2}{1 - \lambda y_2}}\,\frac{\exp(q^{-1}\epsilon K'(y_2))}{(1 - t_2^2)}\,\frac{1 - \mathrm{i}t_2\sqrt{y_2}\,\mathrm{sn}(\mathrm{i}qu, y_2)}{\mathrm{dn}(\mathrm{i}qu, y_2)}\,\mathrm{e}^{-\epsilon u}\,. \tag{32}$$

Similarily, using the first equations of (24) and (25) we obtain, after the necessary transformations, the generating function for the other set of components of the eigenvectors ψ_n

$$\psi(t) = \frac{\mathrm{e}^{w\epsilon}}{\sqrt{t^2 + \lambda - 2ht}} = \sqrt{\frac{1 - y_2^2}{(1 - \lambda y_2)}}\,\frac{\exp(q^{-1}\epsilon K'(y_2))}{(1 - t_2^2)}\,\frac{1 - \mathrm{i}t_2\sqrt{y_2}\,\mathrm{sn}(\mathrm{i}qu, y_2)}{\sqrt{y_2}\,\mathrm{cn}(\mathrm{i}qu, y_2)\,\mathrm{e}^{-\epsilon u}}\,. \tag{33}$$

To simplify the notation we shall set

$$\mathcal{N} = \sqrt{\frac{1 - y_2^2}{(1 - \lambda y_2)}}\,\exp(q^{-1}\epsilon K'(y_2)) \tag{34}$$

and

$$\xi = \mathrm{i}\sqrt{y_2}\,\mathrm{sn}(\mathrm{i}qu, y_2) \tag{35}$$

so that

$$\phi(t) = \frac{\mathcal{N}}{(1 - t_2^2)}\,\frac{1 - t_2\xi}{\mathrm{dn}(\mathrm{i}qu, y_2)}\,\mathrm{e}^{\mathrm{i}q^{-1}\epsilon(\mathrm{i}qu)} \quad \text{and} \quad \psi(t) = \frac{\mathcal{N}}{(1 - t_2^2)}\,\frac{1 - t_2\xi}{\sqrt{y_2}\,\mathrm{cn}(\mathrm{i}qu, y_2)}\,\mathrm{e}^{\mathrm{i}q^{-1}\epsilon(\mathrm{i}qu)}\,. \tag{36}$$

6. The elliptic polynomials ψ_n and ϕ_n

From the generating functions (12) and (13) we may compute the coefficients ψ_n and ϕ_n, which are of course functions of the eigenvalue ϵ, with the help of Cauchy's formula

$$\psi_n = \frac{1}{2\mathrm{i}\pi} \oint \psi(t) t^{-n-1}\,\mathrm{d}t\,.$$

Substituting $\psi(t)$ by its expression (36) and using the variable ξ with

$$t = \frac{t_2 - \xi}{1 - t_2\xi} \qquad \mathrm{d}t = -\frac{(1-t_2^2)}{(1-t_2\xi)^2}\,\mathrm{d}\xi$$

we obtain

$$\psi_n(x) = -\frac{\mathcal{N}}{2\mathrm{i}\pi} \oint \frac{\mathrm{e}^{\mathrm{i}xz}}{\sqrt{y_2}\,\mathrm{cn}(z, y_2)} \frac{(1-t_2\xi)^n}{(t_2-\xi)^{n+1}}\,\mathrm{d}\xi \tag{37}$$

where $z = \mathrm{i}qu$ and $x = q^{-1}\epsilon$, the scaled eigenvalue.

We can express ψ_n in terms of the usual Carlitz polynomials (cf [10])

$$\frac{\mathrm{e}^{\mathrm{i}xz}}{\mathrm{cn}(z, y_2)} = \sum_{p=0}^{\infty} \left(\frac{\xi}{\sqrt{y_2}}\right)^p \frac{P_p^\star(x)}{p!} \tag{38}$$

where

$$P_p^\star(x) = \begin{cases} C_p^\star & \text{for } p \text{ even} \\ D_p^\star & \text{for } p \text{ odd} \end{cases}$$

are the Carlitz polynomials of even and odd order, respectively, associated with the Jacobian elliptic functions cn and dn. Then ψ_n takes the form of a formally infinite series in these polynomials:

$$\psi_n(x) = -\frac{\mathcal{N}}{2\mathrm{i}\pi} \sum_{p=0}^{\infty} \frac{P_p^\star(x)}{(\sqrt{y_2})^{p+1}p!} \oint \xi^p \frac{(1-t_2\xi)^n}{(t_2-\xi)^{n+1}}\,\mathrm{d}\xi\,. \tag{39}$$

The condition $t = 0$ is equivalent to $\zeta = t_2 - \xi = 0$. Thus the contour integral in (39) may be evaluated directly in the complex ζ plane

$$-\oint (t_2 - \zeta)^p \frac{(1 - t_2^2 + t_2\zeta)^n}{\zeta^{n+1}}\,\mathrm{d}\zeta = 2\mathrm{i}\pi\sigma_n(p)$$

where

$$\sigma_n(p) = \sum_{q+m=p} (-1)^{p-q} \binom{p}{q}\binom{n}{m} (1-t_2^2)^m (t_2)^{n-m+q}\,. \tag{40}$$

We arrive at

$$\psi_n(x) = \mathcal{N} \sum_{p=0}^{\infty} \sigma_n(p) \frac{P_p^\star(x)}{(\sqrt{y_2})^{p+1}p!}\,. \tag{41}$$

However, the $\psi_n(x)$ are nevertheless polynomials of finite order n in the variable x. To see this we first evaluate (41) formally for $n = 0$. We have

$$\sigma_0(p) = (t_2)^p \qquad \text{and} \qquad \psi_0(x) = \frac{\mathcal{N}}{\sqrt{y_2}} \sum_{p=0}^{\infty} \left(\frac{t_2}{\sqrt{y_2}} \right)^p \frac{P_p^*(x)}{p!} .$$

Defining z_2 by

$$t_2 = \mathrm{i}\sqrt{y_2}\,\mathrm{sn}(z_2, y_2)$$

we can resume the series and find

$$\psi_0(x) = \frac{\mathcal{N}}{\sqrt{y_2}} \frac{\mathrm{e}^{\mathrm{i}xz_2}}{\mathrm{cn}(z_2, y_2)} \tag{42}$$

which is essentially a constant. From ψ_0 taken formally at arbitrary values of z we can compute all ψ_n for $n > 0$ and thereby indeed show that ψ_n are polynomials of finite order n. For ψ_1 we obtain, for example, the following formula:

$$\psi_1(x) = \left\{ t_2 - (1 - t_2^2) \frac{\partial}{\partial t_2} \right\} \psi_0(x)$$

which is a polynomial in x of first order after dividing out the exponential.

More generally we can show that ψ_n is a polynomial of order n in x given explicitly by

$$\begin{aligned} \psi_n(x) = \Bigg\{ t_2^n - \binom{n}{n-1} t_2^{n-1} \frac{(1-t_2^2)}{1!} \frac{\partial}{\partial t_2} + \binom{n}{n-2} t_2^{n-2} \frac{(1-t_2^2)^2}{2!} \frac{\partial^2}{\partial t_2^2} + \\ \cdots + \frac{(-1)^q}{q!} \binom{n}{n-q} t_2^{n-q} (1-t_2^2)^q \frac{\partial^q}{\partial t_2^q} + \cdots + \frac{(-1)^n}{n!} \binom{n}{n} (1-t_2^2)^n \frac{\partial^n}{\partial t_2^n} \Bigg\} \psi_0(x) . \end{aligned} \tag{43}$$

With the same procedure we show that ϕ_n are polynomials in x of finite order n

$$\phi_n(x) = \mathcal{N} \sum_{p=0}^{\infty} \sigma_n(p) \frac{Q_p^*(x)}{(\sqrt{y_2})^p p!} . \tag{44}$$

Here

$$Q_p^*(x) = \begin{cases} D_p^* & \text{for } p \text{ even} \\ C_p^* & \text{for } p \text{ odd} \end{cases}$$

and we have for $n = 0$

$$\phi_0(x) = \mathcal{N} \frac{\mathrm{e}^{\mathrm{i}xz_2}}{\mathrm{dn}(z_2, y_2)} \tag{45}$$

from which we again compute all polynomials $\phi_n(x)$ of order $n > 0$ explicitly:

$$\begin{aligned} \phi_n(x) = \Bigg\{ t_2^n - \binom{n}{n-1} t_2^{n-1} \frac{(1-t_2^2)}{1!} \frac{\partial}{\partial t_2} + \binom{n}{n-2} t_2^{n-2} \frac{(1-t_2^2)^2}{2!} \frac{\partial^2}{\partial t_2^2} + \\ \cdots + \frac{(-1)^q}{q!} \binom{n}{n-q} t_2^{n-q} (1-t_2^2)^q \frac{\partial^q}{\partial t_2^q} + \cdots + \frac{(-1)^n}{n!} \binom{n}{n} (1-t_2^2)^n \frac{\partial^n}{\partial t_2^n} \Bigg\} \phi_0(x) . \end{aligned} \tag{46}$$

7. Real integral representation and asymptotic behaviour

In the same spirit as in [10] we now derive the *real* integral representation for $\psi_n(x)$, using (37) in the complex z-plane. As before in [10], we choose the rectangular contour $\mathrm{Im}(z) = \pm K'(y_2)$ and $\mathrm{Re}(z) = \pm K(y_2)$ surrounding the point z_2 which now is not necessarily at 0, as was the case in [10]. We obtain $\psi_n(x)$ as a sum of two contributions of the form

$$\begin{aligned}\psi_n(x) = \frac{\sqrt{y_2}\mathcal{N}}{2\pi}\int_{-K(y_2)}^{K(y_2)}\bigg\{&\mathrm{e}^{xK'(y_2)+\mathrm{i}xv}\frac{(t_2+\mathrm{i}\sqrt{y_2}\,\mathrm{sn}(v,y_2))^n}{(1+\mathrm{i}t_2\sqrt{y_2}\,\mathrm{sn}(v,y_2))^{n+1}}\\ &+\mathrm{e}^{-xK'(y_2)-\mathrm{i}xv}\frac{(t_2-\mathrm{i}\sqrt{y_2}\,\mathrm{sn}(v,y_2))^n}{(1-\mathrm{i}t_2\sqrt{y_2}\,\mathrm{sn}(v,y_2))^{n+1}}\bigg\}\mathrm{cn}(v,y_2)\,\mathrm{d}v\\ &+\frac{\kappa\mathcal{N}}{2\pi}\int_{-K(\kappa)}^{K(\kappa)}\bigg\{\mathrm{e}^{\mathrm{i}xK'(\kappa)-xv}\frac{(t_2\sqrt{y_2}+\mathrm{i}\,\mathrm{dn}(v,\kappa))^n}{(\sqrt{y_2}+\mathrm{i}t_2\,\mathrm{dn}(v,\kappa))^{n+1}}\\ &+\mathrm{e}^{-\mathrm{i}xK'(\kappa)+xv}\frac{(t_2\sqrt{y_2}-\mathrm{i}\,\mathrm{dn}(v,\kappa))^n}{(\sqrt{y_2}-\mathrm{i}t_2\,\mathrm{dn}(v,\kappa))^{n+1}}\bigg\}\mathrm{cn}(v,\kappa)\,\mathrm{d}v\,. \end{aligned}\qquad(47)$$

Note that in the second part of (47) the elliptic modulus $\kappa = \sqrt{1-y_2^2}$, the complementary modulus of y_2, appears. As in [10], the second contribution in (47) dominates when $n \to \infty$ because the complex number

$$\frac{t_2 \pm \mathrm{i}\sqrt{y_2}\,\mathrm{sn}(v,y_2)}{1 \pm \mathrm{i}t_2\sqrt{y_2}\,\mathrm{sn}(v,y_2)}$$

has a modulus smaller than 1:

$$\sqrt{\frac{t_2^2 + y_2\,\mathrm{sn}^2(v,y_2)}{1+t_2^2 y_2\,\mathrm{sn}^2(v,y_2)}} < 1$$

since $y_2\,\mathrm{sn}^2(v,y_2) < 1$ is always fulfilled for general values of v. On the contrary, the modulus of the complex number

$$\frac{t_2\sqrt{y_2} \pm \mathrm{i}\,\mathrm{dn}(v,\kappa)}{\sqrt{y_2} \pm \mathrm{i}t_2\,\mathrm{dn}(v,\kappa)}$$

may be larger than 1, since $y_2 \leqslant \mathrm{dn}(v,\kappa) \leqslant 1$, i.e.

$$\sqrt{\frac{t_2^2 y_2 + \mathrm{dn}^2(v,\kappa)}{y_2 + t_2^2\,\mathrm{dn}^2(v,\kappa)}} \qquad \begin{cases} > 1 & \text{for } 0 < v < K/2 \\ < 1 & \text{for } K/2 < v < K\,. \end{cases}$$

Hence for $n \to \infty$ we only have to consider

$$\begin{aligned}\psi_n(x) = \frac{\kappa\mathcal{N}}{2\pi}\int_{-K(\kappa)}^{K(\kappa)}\bigg\{&\mathrm{e}^{\mathrm{i}xK'(\kappa)-xv}\frac{(t_2\sqrt{y_2}+\mathrm{i}\,\mathrm{dn}(v,\kappa))^n}{(\sqrt{y_2}+\mathrm{i}t_2\,\mathrm{dn}(v,\kappa))^{n+1}}\\ &+\mathrm{e}^{-\mathrm{i}xK'(\kappa)+xv}\frac{(t_2\sqrt{y_2}-\mathrm{i}\,\mathrm{dn}(v,\kappa))^n}{(\sqrt{y_2}-\mathrm{i}t_2\,\mathrm{dn}(v,\kappa))^{n+1}}\bigg\}\mathrm{cn}(v,\kappa)\,\mathrm{d}v\,. \end{aligned}\qquad(48)$$

Using the fact that $\mathrm{dn}(v, \kappa)$ and $\mathrm{cn}(v, \kappa)$ are both even functions of the variable v, we derive an alternative form for the asymptotic of $\psi_n(x)$

$$\psi_n(x) \cong \frac{2\kappa\mathcal{N}}{\pi}\{\cos(xK'(\kappa))\mathcal{I}_n(x) - \sin(xK'(\kappa))\mathcal{I}'_n(x)\} \tag{49}$$

where $\mathcal{I}_n(x)$ and $\mathcal{I}'_n(x)$ are the following integrals:

$$\mathcal{I}_n(x) = \int_0^{K(\kappa)} \frac{\sqrt{y_2}\cos n\varphi + t_2\,\mathrm{dn}(v,\kappa)\sin n\varphi}{y_2 + t_2^2\,\mathrm{dn}^2(v,\kappa)} \cosh(xv)\,\mathrm{cn}(v,\kappa)\rho^n(v)\,\mathrm{d}v \tag{50}$$

$$\mathcal{I}'_n(x) = \int_0^{K(\kappa)} \frac{\sqrt{y_2}\sin n\varphi - t_2\,\mathrm{dn}(v,\kappa)\cos n\varphi}{y_2 + t_2^2\,\mathrm{dn}^2(v,\kappa)} \cosh(xv)\,\mathrm{cn}(v,\kappa)\rho^n(v)\,\mathrm{d}v \tag{51}$$

with $\rho(v)$ and $\varphi(v)$ defined by

$$\rho(v)\,\mathrm{e}^{\mathrm{i}\varphi(v)} = \frac{t_2\sqrt{y_2} + \mathrm{i}\,\mathrm{dn}(v,\kappa)}{\sqrt{y_2} + \mathrm{i}t_2\,\mathrm{dn}(v,\kappa)}\,. \tag{52}$$

We observe that $\rho(v)$ is a decreasing function of v, aquiring values from $\sqrt{(t_2^2y_2+1)/(y_2+t_2^2)}$ at $v = 0$ to $\sqrt{(y_2+t_2^2)/(t_2^2y_2+1)}$ at $v = K(\kappa)$. Since $\rho(\frac{1}{2}K) = 1$, the integration interval for large $n \to \infty$ is practically limited to $[0, K/2]$ only and we can approximate ρ by its maximum value $\rho \cong \sqrt{(t_2^2y_2+1)/(y_2+t_2^2)}$. Moreover, in this interval, $\cosh(xv)\,\mathrm{cn}(v,\kappa) \cong 1$ and peaks at a point near $v = K$. This means that $\mathcal{I}_n(x) \cong \mathcal{I}_n$ and $\mathcal{I}'_n(x) \cong \mathcal{I}'_n$ are constants independent of x:

$$\mathcal{I}_n = \left(\sqrt{\frac{t_2^2y_2+1}{y_2+t_2^2}}\right)^n \int_0^{\frac{1}{2}K(\kappa)} \frac{\sqrt{y_2}\cos n\varphi + t_2\,\mathrm{dn}(v,\kappa)\sin n\varphi}{y_2 + t_2^2\,\mathrm{dn}^2(v,\kappa)}\,\mathrm{d}v \tag{53}$$

$$\mathcal{I}'_n = \left(\sqrt{\frac{t_2^2y_2+1}{y_2+t_2^2}}\right)^n \int_0^{\frac{1}{2}K(\kappa)} \frac{\sqrt{y_2}\sin n\varphi - t_2\,\mathrm{dn}(v,\kappa)\cos n\varphi}{y_2 + t_2^2\,\mathrm{dn}^2(v,\kappa)}\,\mathrm{d}v\,. \tag{54}$$

Within these approximations one may estimate the zeros of $\psi_n(x)$ in the asymptotic limit $n \to \infty$. They are given by the equation

$$\tan(xK'(\kappa)) = \frac{\mathcal{I}_n}{\mathcal{I}'_n} = \tan\theta_n \tag{55}$$

or, for each n, the zeros x_{np} are labelled by p:

$$x_{np} = \theta_n + p\pi\,. \tag{56}$$

The eigenvalue problem set in (2)–(7) is equivalent to

$$\tan(x_{Np}K'(\kappa)) = \frac{\lambda\mathcal{I}_{N+1} - h\mathcal{I}_N}{\lambda\mathcal{I}'_{N+1} - h\mathcal{I}'_N} = \tan\theta_N \tag{57}$$

leading to

$$\epsilon_{Np} = \sqrt{\frac{\lambda - h^2}{y_2}}\frac{1}{K'(\kappa)}\{\theta_N + p\pi\}\,. \tag{58}$$

The spacing between the ϵ_{Np} is thus constant and there is a shift for each N given by θ_N, defined by (57).

A similar analysis for $\phi_n(x)$ may be given. We only quote the results

$$\begin{aligned}\phi_n(x) = \frac{\mathcal{N}}{2\pi}\int_{-K(y_2)}^{K(y_2)} &\left\{ e^{xK'(y_2)+ixv} \frac{(t_2 + i\sqrt{y_2}\,\mathrm{sn}(v, y_2))^n}{(1 + it_2\sqrt{y_2}\,\mathrm{sn}(v, y_2))^{n+1}} \right. \\ &\left. + e^{-xK'(y_2)-ixv} \frac{(t_2 - i\sqrt{y_2}\,\mathrm{sn}(v, y_2))^n}{(1 - it_2\sqrt{y_2}\,\mathrm{sn}(v, y_2))^{n+1}} \right\} \mathrm{dn}(v, y_2)\, dv \\ &+ \frac{i\kappa\mathcal{N}}{2\pi}\int_{-K(\kappa)}^{K(\kappa)} \left\{ -e^{ixK'(\kappa)-xv} \frac{(t_2\sqrt{y_2} + i\,\mathrm{dn}(v, \kappa))^n}{(\sqrt{y_2} + it_2\,\mathrm{dn}(v, \kappa))^{n+1}} \right. \\ &\left. + e^{-ixK'(\kappa)+xv} \frac{(t_2\sqrt{y_2} - i\,\mathrm{dn}(v, \kappa))^n}{(\sqrt{y_2} - it_2\,\mathrm{dn}(v, \kappa))^{n+1}} \right\} \mathrm{sn}(v, \kappa)\, dv\,. \end{aligned} \tag{59}$$

Again only the second integral dominates as $n \to \infty$ for the same reason as before in the expression for ψ_n. This last part may be recast into the following form, which incidently is proportional to i:

$$\phi_n(x) \cong \frac{2i\kappa\mathcal{N}}{\pi}\{\cos(xK'(\kappa))\bar{\mathcal{I}}_n(x) - \sin(xK'(\kappa))\bar{\mathcal{I}}'_n(x)\} \tag{60}$$

with

$$\bar{\mathcal{I}}_n(x) = \int_0^{K(\kappa)} \frac{\sqrt{y_2}\cos n\varphi + t_2\,\mathrm{dn}(v, \kappa)\sin n\varphi}{y_2 + t_2^2\,\mathrm{dn}^2(v, \kappa)} \sinh(xv)\,\mathrm{sn}(v, \kappa)\rho^n(v)\, dv \tag{61}$$

$$\bar{\mathcal{I}}'_n(x) = \int_0^{K(\kappa)} \frac{\sqrt{y_2}\sin n\varphi - t_2\,\mathrm{dn}(v, \kappa)\cos n\varphi}{y_2 + t_2^2\,\mathrm{dn}^2(v, \kappa)} \sinh(xv)\,\mathrm{sn}(v, \kappa)\rho^n(v)\, dv \tag{62}$$

where ρ and φ are again given by (52). For $n \to \infty$ the behaviour of ρ again limits the integration to the interval $[0, K/2]$ and ρ can be approximated by its maximum value $\rho \cong \sqrt{(t_2^2 y_2 + 1)(y_2 + t_2^2)}$. However, here the product $\sinh(xv)\,\mathrm{sn}(v, \kappa)$ behaves as xv^2 in $[0, K/2]$, instead of being nearly a constant of order unity. Thus we obtain the asymptotic behaviour for $\phi_n(x)$

$$\phi_n(x) \cong \frac{2i\kappa\mathcal{N}}{\pi}\{x\cos(xK'(\kappa))\bar{\mathcal{I}}_n - x\sin(xK'(\kappa))\bar{\mathcal{I}}'_n\} \tag{63}$$

where now

$$\bar{\mathcal{I}}_n = \left(\sqrt{\frac{t_2^2 y_2 + 1}{y_2 + t_2^2}}\right)^n \int_0^{\frac{1}{2}K(\kappa)} v^2 \frac{\sqrt{y_2}\cos n\varphi + t_2\,\mathrm{dn}(v, \kappa)\sin n\varphi}{y_2 + t_2^2\,\mathrm{dn}^2(v, \kappa)}\, dv$$

$$\bar{\mathcal{I}}'_n = \left(\sqrt{\frac{t_2^2 y_2 + 1}{y_2 + t_2^2}}\right)^n \int_0^{\frac{1}{2}K(\kappa)} v^2 \frac{\sqrt{y_2}\sin n\varphi - t_2\,\mathrm{dn}(v, \kappa)\cos n\varphi}{y_2 + t_2^2\,\mathrm{dn}^2(v, \kappa)}\, dv$$

are constants independent of x. The zeros x_{np} of $\phi_n(x)$ are asymptotically given by

$$\tan(x_{np}K'(\kappa)) = \frac{\bar{\mathcal{I}}_n}{\bar{\mathcal{I}}'_n} = \tan\bar{\theta}_n\,. \tag{64}$$

The boundary conditions (6) and (7) of the CTM problem yield the eigenvalues ϵ_{Np}

$$\tan(q^{-1}\epsilon_{Np}K'(\kappa)) = \tan\bar{\theta}_N = \frac{\bar{\mathcal{I}}_{N+1} - h\bar{\mathcal{I}}_N}{\bar{\mathcal{I}}'_{N+1} - h\bar{\mathcal{I}}'_N} \tag{65}$$

or

$$\epsilon_{Np} = \frac{q}{K'(\kappa)}\{\bar{\theta}_N + p\pi\}. \tag{66}$$

We observe again that the levels ϵ_{Np} are equidistant in this limit of large N, but there is a translation by an amount $(q/K'(\kappa))\bar{\theta}_N$ for each polynomial of order N.

Equations (58) and (66) are the main results of this paper, again confirming, for the simplest vertex model, the results of Baxter [2] in an explicit calculation for a finite system.

A last remark concerns the level spacing. From (58) and (66) we have

$$\Delta\epsilon = \epsilon_{N,p+1} - \epsilon_{N,p} = \frac{q\pi}{K'(\kappa)} = \frac{q\pi}{K(y_2)}. \tag{67}$$

The modulus used here is y_2 of (22). For $h = 0$ we have $y_2 = \lambda$, which is thus different from the modulus parametrization used in [10], which was λ^{-1}. Moreover, the normalization of the energy levels is also different due to the fact that the Carlitz polynomials are used directly in [10]. The Carlitz polynomials are normalized so that the first one is always 1 or x, depending on the parity. Here our $\psi_0(x)$ is not 1 but contains an x dependence according to (42) which may be divided out later.

At $h = 0$, where $y_2 = \lambda$ and $t_2 = 0$, there is a decoupling in (47) and (59), respectively, and we recover the results of [10].

As in [1], we have checked the level spacing (67) numerically with standard methods (cf [20]) of diagonalizing the pentadiagonal matrix which is equivalent to the recursion relations (2) and (3) if either ψ_n or ϕ_n is eliminated. We have observed equidistant level spacing to rather high accuracy already for very moderate system sizes of the order of $N = 20$.

8. Summary and conclusion

The generator of the CTM of a generalized free-Fermion vertex system of finite size is a quantum spin chain Hamiltonian with particular interactions which increase linearly along the chain. We have presented the analytical diagonalization of this particular quantum spin chain in the asymptotic regime of large system size N for arbitrary values of the parameters, the anisotropy λ and the magnetic field h, in the region where $\lambda > h^2$.

Let us briefly summarize the methods applied to accomplish our goal and restate our main result for easy reference. The asymptotic diagonalization has been achieved through the explicit construction of a new class of elliptic polynomials which are the components of the eigenvectors of the problem. In this construction an elliptic parametrization of the generating functions of the polynomials has been used which is based on the treatment of a two-parameter elliptic integral. The asymptotic evaluation of an integral representation of these polynomials yields the eigenvalues, given by (58) and (66), respectively, which are equidistantly spaced with spacing

$$\Delta\epsilon = \frac{q\pi}{K(y_2)}$$

the modulus y_2 of the complete elliptic integral $K(y_2)$ being related to the generating functions of the eigenvectors and given explicitly in (22). This equidistant spacing is the main result of the present work, extending the findings of previous studies [1, 7, 9] to general values of the parameters and thereby confirming once again the general expectation [2].

We have not touched on the issue of the orthogonality of the polynomials $\psi_n(x)$ and $\phi_n(x)$, which may be called associate Carlitz polynomials. Since the three limiting cases are orthogonal polynomials, it is natural to expect that the 'associated Carlitz polynomials' remain orthogonal. Presumably the proof is based on the continuous fraction expansion of some elliptic functions interpolating between the Jacobian elliptic functions $\mathrm{cn}(x, k)$ and $\mathrm{dn}(x, k)$ [14]. We have not succeded in proving this yet.

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Appendix

As we have seen, the main problem has been the inversion of the elliptic integral of (15). The solution obtained was given in (31), which, upon replacing t_2 by its expression, reads

$$t = \frac{h(1-y_2) - \mathrm{i}(1-\lambda y_2)\sqrt{y_2}\,\mathrm{sn}(\mathrm{i}qu, y_2)}{(1-\lambda y_2) - \mathrm{i}h(1-y_2)\sqrt{y_2}\,\mathrm{sn}(\mathrm{i}qu, y_2)}.$$

In this appendix we check the limit $\lambda \to 1$ against a direct calculation. Let us assume $\lambda = 1-\epsilon$, then $y_2 = 1-\epsilon/\sqrt{1-h^2}$ or, using $h = \cos\theta$ as in [7], $y_2 = 1-\epsilon/\sin\theta$. Then to first order in ϵ, $(1-\lambda y_2) \cong \epsilon(1+1/\sin\theta)$, and we have

$$\lim_{\lambda\to 1} \mathrm{sn}(\mathrm{i}qu, y_2) = \tanh(\mathrm{i}u\sin\theta)$$

and

$$\lim_{\lambda\to 1} t = \frac{\cos\theta + (1+\sin\theta)\tan(u\sin\theta)}{(1+\sin\theta) + \cos\theta\tan(u\sin\theta)}$$

which upon inversion gives u as a function of t:

$$u(t) = \frac{1}{2\mathrm{i}\sin\theta}\ln\left(\frac{(1+\sin\theta - t\cos\theta) + \mathrm{i}(\cos\theta - (1+\sin\theta)t)}{(1+\sin\theta - t\cos\theta) - \mathrm{i}(\cos\theta - (1+\sin\theta)t)}\right)$$

or

$$u(t) = \frac{1}{2\mathrm{i}\sin\theta}\ln\left(\frac{t-\mathrm{e}^{-\mathrm{i}\theta}}{t-\mathrm{e}^{\mathrm{i}\theta}}\right) + \frac{\theta+\pi/2}{2\sin\theta}$$

which agrees with the $u(t)$ computed directly from the integral (15) with $\lambda = 1$.

Next we check the limit $\lambda \to h^2$. There we have

$$y_2 \cong \frac{\lambda - h^2}{(1-h^2)^2} \to 0$$

and

$$\lim_{y_2 \to 0} \mathrm{i}\sqrt{y_2}\,\mathrm{sn}(\mathrm{i}qu, y_2) = -\frac{\sqrt{\lambda - h^2}}{(1-h^2)} \sinh((1-h^2)u)\,.$$

Now to obtain the correct limit we must impose a shift

$$u(1-h^2) = x(1-h^2) - \tfrac{1}{2}K'(y_2)\,.$$

Then as $y_2 \to 0$

$$\lim_{y_2 \to 0} = x(1-h^2) - \tfrac{1}{2}\ln(4/y_2) = x(1-h^2) - \ln\left(\sqrt{\frac{4(1-h^2)^2}{\lambda - h^2}}\right)$$

leads to

$$\sinh((1-h^2)u) \cong -\frac{1}{2}\frac{2(1-h^2)}{\sqrt{\lambda - h^2}}\,\mathrm{e}^{-x(1-h^2)}\,.$$

Finally for $\lambda \to h^2$

$$\lim_{y_2 \to 0} \mathrm{i}\sqrt{y_2}\,\mathrm{sn}(\mathrm{i}qu, y_2) \doteq \mathrm{e}^{-x(1-h^2)}$$

which one obtains by direct integration.

The correctness of the two limits implies that the limiting generating functions are generating functions for Meixner polynomials of the first and second kinds according to the classification of Chihara [18] or the Gottlieb and Meixner Pollaczek polynomials according to an independent classification.

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